MATH 54 - MOCK FINAL EXAM - SOLUTIONS

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1. *(20 = 15 + 5 points)*
   (a) Find a diagonal matrix D and an orthogonal matrix P such that $A = PDP^T$, where:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

<u>Eigenvalues:</u>

$$\begin{aligned} det(A - \lambda I) = & \begin{vmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} \\ =&(3-\lambda)\begin{vmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} -2 & 6-\lambda \\ 4 & 2 \end{vmatrix} \\ =&(3-\lambda)\left((6-\lambda)(3-\lambda)-4\right) + 2\left((-2)(3-\lambda)-8\right) + 4\left(-4-4(6-\lambda)\right) \\ =&-\lambda^3 + 12\lambda^2 - 21\lambda - 98 \end{aligned}$$

Now, using the rational roots theorem (the possible zeros are $\pm1, \pm2, \pm7, \pm14, \pm49, \pm98$), we get that $\lambda = -2$ is an eigenvalue, and using long division, we get:

$$-\lambda^3+12\lambda^2-21\lambda-98 = -(\lambda-2)(\lambda^2-14\lambda+49) = -(\lambda-2)(\lambda-7)^2 = 0$$

Hence the eigenvalues are $\boxed{\lambda = -2, 7}$

<u>Eigenvectors</u>

<u>$\lambda = -2$</u>:

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$$Nul(A+2I) = Nul \begin{bmatrix} 5 & 2 & -4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} = Nul \begin{bmatrix} 5 & -2 & 4 \\ 1 & -4 & -1 \\ 0 & 18 & 9 \end{bmatrix}$$

$$= Nul \begin{bmatrix} 1 & -4 & -1 \\ 0 & 18 & 9 \\ 0 & 18 & 9 \end{bmatrix} = Nul \begin{bmatrix} 1 & -4 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= Nul \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$= Span \left\{ \begin{bmatrix} -1 \\ \frac{-1}{2} \\ 1 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\}$$

$\underline{\lambda = 7}$:

$$Nul(A-7I) = Nul \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} = Nul \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

$$= Nul \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= Span \left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

<u>Gram-Schmidt</u>

$\underline{\lambda = -2}$:

Let $\mathbf{u_1} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, then $\mathbf{v_1} = \mathbf{u_1}$, and:

$$\mathbf{w_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{bmatrix}$$

$\underline{\lambda = 7}$:

Let $\mathbf{u_1} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Then:

$$\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{u_2} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix} \sim \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

And finally:

$$\mathbf{w_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \\ 0 \end{bmatrix}, \mathbf{w_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} = \begin{bmatrix} \frac{4}{\sqrt{45}} \\ \frac{2}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \end{bmatrix}$$

<u>Conclusion:</u>
Putting everything together, we get $A = PDP^T$, where:

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, P = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ \frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ \frac{-2}{3} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix}$$

(b) Use (a) to write the quadratic form $3x_1^2 + 6x_2^2 + 3x_3^2 - 4x_1x_2 + 8x_1x_3 + 4x_2x_3$ without cross-product terms.

The matrix of the quadratic form is:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Hence, if you define $\mathbf{x} = P\mathbf{y}$, that is $\mathbf{y} = P^T\mathbf{x}$, where P is as above, then the quadratic form becomes:

$$-2y_1^2 + 7y_2^2 + 7y_3^2$$

2. *(20 points, 2 points each)*

Mark the following statements as **TRUE** or **FALSE**. If the statement is **TRUE**, don't do anything. If the statement is **FALSE**, provide an explicit counterexample.

(a) If A is a 3×3 matrix with eigenvalues $\lambda = 0, 2, 3$, then A must be diagonalizable!

TRUE (an $n \times n$ matrix with 3 distinct eigenvalues is diagonalizable)

(b) There does not exist a 3×3 matrix A with eigenvalues $\lambda = 1, -1, -1 + i$.

TRUE (here we assume A has real entries; eigenvalues always come in complex conjugate pairs, i.e. if A has eigenvalue $-1 + i$, it must also have eigenvalue $-1 - i$)

(c) If A is a symmetric matrix, then all its eigenvectors are orthogonal.

FALSE: Take A to be your favorite symmetric matrix, and, for example, take $\mathbf{v}$ to be one eigenvector, and $\mathbf{w}$ to be the *same* eigenvector (or a different eigenvector corresponding to the same eigenvalue). That's why we had to apply the Gram Schmidt process to each eigenspace in the previous problem!

(d) If Q is an orthogonal $n \times n$ matrix, then $Row(Q) = Col(Q)$.

TRUE: (since Q is orthogonal, $Q^T Q = I$, so Q is invertible, hence $Row(Q) = Col(Q) = \mathbb{R}^n$)

(e) The equation $A\mathbf{x} = \mathbf{b}$, where A is a $n \times n$ matrix always has a unique least-squares solution.

FALSE: Take A to be the zero matrix, and $\mathbf{b}$ to be the zero vector! This statement is true if A has rank n.

(f) If $AB = I$, then $BA = I$.

FALSE: Let $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $AB = I$, but BA isn't even defined!

(g) If A is a square matrix, then $Rank(A) = Rank(A^2)$

FALSE: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $Rank(A) = 1$, but $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $Rank(A^2) = 0$.

(h) If W is a subspace, and $P\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto W, then $P^2\mathbf{y} = P\mathbf{y}$

TRUE (draw a picture! If you orthogonally project $P\mathbf{y} = \hat{\mathbf{y}}$ on W, you get $\hat{\mathbf{y}}$)

(i) If $T : V \to W$, where $dim(V) = 3$ and $dim(W) = 2$, then T cannot be one-to-one.

TRUE (by Rank-Nullity theorem, $dim(Nul(T)) + Rank(T) = 3$. But $Rank(T)$ can only be at most $dim(W) = 2$, so $dim(Nul(T)) > 0$, so $Nul(T) \neq \{\mathbf{0}\}$)

(j) If A is similar to B, then $det(A) = det(B)$.

TRUE (If $A = PBP^{-1}$, then $det(A) = det(B)$)

3. *(20 points)* Solve the following system $\mathbf{x}' = A\mathbf{x}$, where:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

<u>Eigenvalues:</u>

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((1-\lambda)^2 + 1\right) = 0$$

which gives you $\lambda = 1, 1 \pm i$.

<u>Eigenvectors:</u>

$\underline{\lambda = 1}$:

$$Nul(A - I) = Nul \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = Nul \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = Span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\underline{\lambda = 1 + i}$

$$\begin{aligned} Nul(A - (1+i)I) &= Nul \begin{bmatrix} -i & 2 & -1 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix} \\ &= Nul \begin{bmatrix} 1 & 2i & -i \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \\ &= Nul \begin{bmatrix} 1 & 0 & 2-i \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \\ &= Span \left\{ \begin{bmatrix} -2+i \\ i \\ 1 \end{bmatrix} \right\} \end{aligned}$$

Note: Here we used the fact that $\frac{1}{i} = \frac{i}{i^2} = -i$.

Now separate the eigenvector into real and imaginary parts:

$$\begin{bmatrix} -2+i \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

<u>Solution:</u> Hence, by the formula on page 598 of the book, our solution is:

$$\mathbf{x}(t) = Ae^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + B\left(e^t\cos(t)\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - e^t\sin(t)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) + C\left(e^t\sin(t)\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + e^t\cos(t)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

4. *(10 points)* Solve the following system $\mathbf{x}' = A\mathbf{x}$, where:

$$A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$$

<ins>Eigenvalues:</ins>

$$det(A-\lambda I) = \begin{bmatrix} 1-\lambda & -1 \\ 4 & -3-\lambda \end{bmatrix} = (1-\lambda)(-3-\lambda)+4 = \lambda^2+2\lambda+1 = (\lambda+1)^2 = 0$$

which gives $\lambda = -1$.

<ins>Eigenvectors</ins>

$$Nul(A+I) = Nul \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} = Nul \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = Span \left\{ \begin{bmatrix} 1 & 2 \end{bmatrix} \right\}$$

<ins>Generalized eigenvector:</ins>

Now find $\mathbf{u}$ such that $(A+I)\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$$\begin{bmatrix} 2 & -1 & | \; 1 \\ 4 & -2 & | \; 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & | \; 1 \\ 0 & 0 & | \; 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid s \in \mathbb{R} \right\}$$

Now let $s = 0$, and you get $\mathbf{u} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

<ins>Solution:</ins>

$$\mathbf{x}(t) = Ae^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + B \left(te^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

5. *(15 points)* Assume you're given a coupled mass/spring system with $N = 3$, $m_1 = m_2 = m_3 = 1$ and $k_1 = k_2 = k_3 = k_4 = 1$. Find the proper frequencies and proper modes.

Equation:

$\mathbf{x}'' = A\mathbf{x}$, where:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Proper frequencies:

Find the eigenvalues of the matrix A: $\lambda = -2, -2-\sqrt{2}, -2+\sqrt{2}$

Fact: The proper frequencies are $\pm\sqrt{\lambda}$.

Hence the proper frequencies are:

$$\pm\sqrt{-2} = \pm\sqrt{2}i, \quad \pm\sqrt{-2-\sqrt{2}} = \pm\left(\sqrt{2+\sqrt{2}}\right)i, \quad \pm\sqrt{-2+\sqrt{2}} = \pm\left(\sqrt{2-\sqrt{2}}\right)i$$

Proper modes:

To find the modes, use the following trick: Since $N = 3$, $N + 1 = 4$, hence all the modes will involve $\sin\left(q\frac{\pi}{4}\right)$, where q is some number. Then:

$$\mathbf{v_1} = \begin{bmatrix} \sin\left(1\frac{\pi}{4}\right) \\ \sin\left(2\frac{\pi}{4}\right) \\ \sin\left(3\frac{\pi}{4}\right) \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} \sin\left(2\frac{\pi}{4}\right) \\ \sin\left(4\frac{\pi}{4}\right) \\ \sin\left(6\frac{\pi}{4}\right) \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} \sin\left(3\frac{\pi}{4}\right) \\ \sin\left(6\frac{\pi}{4}\right) \\ \sin\left(9\frac{\pi}{4}\right) \end{bmatrix}$$

Note: The way you get the other values is by using multiples, i.e. the multiples of 1 are 1, 2 and 3, the multiples of 2 are 2, 4 and 6, the multiples of 3 are 3, 6, and 9.

Hence the proper modes are:

$$\mathbf{v_1} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -1 \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

6. *(20 points)* Solve the following heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & 0 < x < 1, \quad t > 0 \\ u(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0 & t > 0 \\ u(x,0) = x & 0 < x < 1 \end{cases}$$

Step 1: Separation of variables. Suppose:

$$u(x,t) = X(x)T(t) \tag{1}$$

Plug (1) into the differential equation (), and you get:

$$\begin{aligned} (X(x)T(t))_t &= (X(x)T(t))_{tt} \\ X(x)T'(t) &= X''(x)T(t) \end{aligned}$$

Rearrange and get:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} \tag{2}$$

Now $\frac{X''(x)}{X(x)}$ *only* depends on x, but by (2) *only* depends on t, hence it is constant:

$$\begin{aligned} \frac{X''(x)}{X(x)} &= \lambda \\ X''(x) &= \lambda X(x) \end{aligned} \tag{3}$$

Also, we get:

$$\begin{aligned} \frac{T'(t)}{T(t)} &= \lambda \\ T'(t) &= \lambda T(t) \end{aligned} \tag{4}$$

but we'll only deal with that later (Step 4)

Step 2: Consider (3):

$$X''(x) = \lambda X(x)$$

Note: Always start with $X(x)$, do **NOT** touch $T(t)$ until right at the end!

Now use the **boundary conditions** in ():

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u(1,t) = X(1)T(t) = 0 \Rightarrow X(1)T(t) = 0 \Rightarrow X(1) = 0$$

Hence we get:

$$\begin{cases} X''(x) = & \lambda X(x) \\ X(0) = & 0 \\ X(1) = & 0 \end{cases} \tag{5}$$

Step 3: Eigenvalues/Eigenfunctions. The auxiliary polynomial of (5) is $p(\lambda) = r^2 - \lambda$

Now we need to consider 3 cases:

<u>Case 1:</u> $\lambda > 0$, then $\lambda = \omega^2$, where $\omega > 0$

Then:

$$r^2 - \lambda = 0 \Rightarrow r^2 - \omega^2 = 0 \Rightarrow r = \pm\omega$$

Therefore:

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$

Now use $X(0) = 0$ and $X(1) = 0$:

$$X(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A \Rightarrow X(x) = Ae^{\omega x} - Ae^{-\omega x}$$

$$X(1) = 0 \Rightarrow Ae^{\omega} - Ae^{-\omega} = 0 \Rightarrow Ae^{\omega} = Ae^{-\omega} \Rightarrow e^{\omega} = e^{-\omega} \Rightarrow \omega = -\omega \Rightarrow \omega = 0$$

But this is a **contradiction**, as we want $\omega > 0$.

Case 2: $\lambda = 0$, then $r = 0$, and:

$$X(x) = Ae^{0x} + Bxe^{0x} = A + Bx$$

And:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = Bx$$

$$X(1) = 0 \Rightarrow B = 0 \Rightarrow X(x) = 0$$

Again, a **contradiction** (we want $X \not\equiv 0$, because otherwise $u(x,t) \equiv 0$)

Case 3: $\lambda < 0$, then $\lambda = -\omega^2$, and:

$$r^2 - \lambda = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm\omega i$$

Which gives:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

Again, using $X(0) = 0$, $X(1) = 0$, we get:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B\sin(\omega x)$$

$$X(1) = 0 \Rightarrow B\sin(\omega) = 0 \Rightarrow \sin(\omega) = 0 \Rightarrow \omega = \pi m, \quad (m = 1, 2, \cdots)$$

This tells us that:

$$\begin{aligned} &\text{Eigenvalues:} \lambda = -\omega^2 = -(\pi m)^2 \quad (m = 1, 2, \cdots) \\ &\text{Eigenfunctions:} X(x) = \sin(\omega x) = \sin(\pi m x) \end{aligned} \tag{6}$$

Step 4: Deal with (4), and remember that $\lambda = -(\pi m)^2$:

$$T'(t) = \lambda T(t) \Rightarrow T(t) = Ae^{\lambda t} = T(t) = \widetilde{A_m} e^{-(\pi m)^2 t} \qquad m = 1, 2, \cdots$$

Note: Here we use $\widetilde{A_m}$ to emphasize that $\widetilde{A_m}$ depends on m.

Step 5: Take linear combinations:

$$u(x,t) = \sum_{m=1}^{\infty} T(t)X(x) = \sum_{m=1}^{\infty} \widetilde{A_m} e^{-(\pi m)^2 t} \sin(\pi m x) \tag{7}$$

Step 6: Use the initial condition $u(x,0) = x$ in ():

$$u(x,0) = \sum_{m=1}^{\infty} \widetilde{A_m} \sin(\pi m x) = x \qquad \text{on}(0,1) \tag{8}$$

Now we want to express x as a linear combination of sines, so we have to use a **sine series** (that's why we used $\widetilde{A_m}$ instead of A_m):

$$\begin{aligned}
\widetilde{A_m} &= \frac{2}{1}\int_0^1 x \sin(\pi m x) dx \\
&= 2\left(\left[-x\frac{\cos(\pi m x)}{\pi m}\right]_0^1 - \int_0^1 -\frac{\cos(\pi m x)}{\pi m} dx\right) \\
&= 2\left(-\frac{\cos(\pi m)}{\pi m} + \int_0^1 \frac{\cos(\pi m x)}{\pi m} dx\right) \\
&= 2\left(-\frac{(-1)^m}{\pi m} + \left[\frac{\sin(\pi m x)}{(\pi m)^2}\right]_0^1\right) \\
&= \frac{2(-1)^{m+1}}{\pi m} \qquad (m = 1, 2, \cdots)
\end{aligned}$$

Step 7: Conclude using (9)

$$u(x,t) = \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{\pi m} e^{-(\pi m)^2 t} \sin(\pi m x) \tag{9}$$

7. *(10 points)* Solve the following wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = 16\dfrac{\partial^2 u}{\partial x^2} & -\infty < x < \infty, \quad t > 0 \\ u(x,0) = e^{-x^2} & -\infty < x < \infty \\ \dfrac{\partial u}{\partial t}(x,0) = \sin(x) & -\infty < x < \infty \end{cases}$$

Hint: Careful! Do not use separation of variables for this, because $-\infty < x < \infty$. Use d'Alembert's formula!

By d'Alembert's formula on page 684 in the book (formula (32)), we get:

$$u(x,t) = \frac{1}{2}(f(x+\alpha t) + f(x-\alpha t)) + \frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} g(s)ds$$

Here $\alpha = \sqrt{16} = 4$, and $f(x) = u(x,0) = e^{-x^2}$, $g(x) = \frac{\partial u}{\partial t}(x,0) = \sin(x)$.

Moreover, $\int \sin(s) = -\cos(s)$, hence:

$$u(x,t) = \frac{1}{2}\left(e^{-(x+4t)^2} + e^{-(x-4t)^2}\right) + \frac{1}{8}\left(\cos(x-4t) - \cos(x+4t)\right)$$

8. *(10 points)* Solve the following equation using either undetermined coefficients or variation of parameters:

$$y'' + 2y' + y = e^t$$

Homogeneous equation:

Aux: $r^2 + 2r + 1 = (r+1)^2 = 0$, which gives $r = -1$ (double root). Hence:

$$y_0(t) = Ae^{-t} + Bte^{-t}$$

Inhomogeneous equation:

Undetermined coefficients:

Guess $y_p(t) = Ae^t$ (that's ok, because the root $r = -1$ doesn't coincide with the inhomogeneous term e^t)

Then, if you plug in y_p into the differential equation, you get:

$$\begin{aligned} Ae^t + 2Ae^t + Ae^t &= e^t \\ 4Ae^t &= e^t \\ 4A &= 1 \\ A &= \frac{1}{4} \end{aligned}$$

Therefore $y_p(t) = \frac{1}{4}e^t$, and:

$$y(t) = Ae^{-t} + Bte^{-t} + \frac{1}{4}e^t$$

Variation of parameters:

First make sure the coefficient of y'' is 1. Check.

Suppose $y_p(t) = v_1(t)e^{-t} + v_2(t)te^{-t}$.

Then let $\widetilde{W}(t)$ be the Wronskian matrix:

$$\widetilde{W}(t) = \begin{bmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

Notice that:

$$det(\widetilde{W})(t) = e^{-t}(e^{-t} - te^{-t}) + e^{-t}(te^{-t}) = e^{-2t} - te^{-2t} + te^{-2t} = e^{-2t}$$

Hence:

$$\left(\widetilde{W}(t)\right)^{-1} = e^{2t} \begin{bmatrix} e^{-t} - te^{-t} & -te^{-t} \\ e^{-t} & e^{-t} \end{bmatrix} = \begin{bmatrix} e^{t} - te^{t} & -te^{t} \\ e^{t} & e^{t} \end{bmatrix}$$

Now solve the following equation:

$$\left(\widetilde{W}(t)\right) \begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$$

which gives:

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \left(\widetilde{W}(t)\right)^{-1} \begin{bmatrix} 0 \\ e^t \end{bmatrix} = \begin{bmatrix} e^{t} - te^{t} & -te^{t} \\ e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} 0 \\ e^t \end{bmatrix} = \begin{bmatrix} -te^{2t} \\ e^{2t} \end{bmatrix}$$

Hence $v_1'(t) = -te^{2t}$, so $v_1(t) = \left(-\frac{t}{2} + \frac{1}{4}\right) e^{2t}$

And $v_2'(t) = e^{2t}$, so $v_2(t) = \frac{e^{2t}}{2}$.

Finally:

$$y_p(t) = v_1(t)e^{-t} + v_2(t)te^{-t} = \left(-\frac{t}{2} + \frac{1}{4}\right) e^t + \frac{t}{2}e^t = \frac{1}{4}e^t$$

And hence:

$$y(t) = Ae^{-t} + Bte^{-t} + te^{t} + \frac{1}{4}e^{t}$$

Bonus *(5 points)*

Disclaimer: This problem is slightly harder than the other ones. It's just meant for the people who're bored and want an extra challenge! Only attempt it if you truly understand vector spaces and linear transformations!

Let V be the vector space of infinitely differentiable functions f from $\mathbb{R}$ to $\mathbb{R}$.

Define $T : V \to V$ by: $T(y) = y'' - 3y' + 2y$.

(a) Show T is a linear transformation.

$$T(y_1+y_2) = (y_1+y_2)''-3(y_1+y_2)'+2(y_1+y_2) = (y_1'' - 3y_1' + 2y_1)+(y_2'' - 3y_2' + 2y_2) = T(y_1)+T(y_2)$$

$$T(cy) = (cy)'' - 3(cy)' + 2(cy) = c(y'' - 3y' + 2y) = cT(y)$$

(b) Find $Nul(T)$.

$Nul(T)$ is the set of y such that $T(y) = 0$, that is $y''-3y'+2y = 0$.

The auxiliary polynomial is $r^2 - 3r + 2 = (r-1)(r-2) = 0$, which gives $r = 1$, $r = 2$, so:

$$y(t) = Ae^t + Be^{2t}$$

which says that:

$$Nul(T) = Span\left\{e^t, e^{2t}\right\}$$

(c) Is T one-to-one?

No, since $Nul(T) \neq \{0\}$

(d) Show T is onto. Namely, given f in V, show that $T(y) = f$ has at least one solution.

Fix f, and we need to find a y such that $T(y) = f$, that is:

$$y'' - 3y' + 2y = f$$

So all we need to do is to find a *particular* solution y to this differential equation!

But notice that by using variation of parameters, we can actually give an explicit formula for y !

In (b), we found that the homogeneous solution is $y(t) = Ae^{-t} + Be^{-2t}$.

<u>Variation of parameters:</u>

Now suppose $y(t) = v_1(t)e^{-t} + v_2(t)e^{-2t}$.

Define the Wronskian matrix:

$$\widetilde{W}(t) = \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix}$$

Then $det(\widetilde{W}(t)) = -e^{-3t}$.

Hence:

$$\left(\widetilde{W}(t)\right)^{-1} = -e^{3t} \begin{bmatrix} -2e^{-2t} & -e^{-2t} \\ e^{-t} & e^{-t} \end{bmatrix} = \begin{bmatrix} 2e^{t} & e^{t} \\ -e^{2t} & -e^{2t} \end{bmatrix}$$

Now solve the following equation:

$$\left(\widetilde{W}(t)\right)\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

which gives:

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \left(\widetilde{W}(t)\right)^{-1}\begin{bmatrix} 0 \\ f(t) \end{bmatrix} = \begin{bmatrix} 2e^t & e^t \\ -e^{2t} & -e^{2t} \end{bmatrix}\begin{bmatrix} 0 \\ f(t) \end{bmatrix} = \begin{bmatrix} e^t f(t) \\ -e^{2t} f(t) \end{bmatrix}$$

Hence $v_1'(t) = e^t f(t)$, so $v_1(t) = \int e^t f(t)dt$

And $v_2'(t) = -e^{2t} f(t)$, so $v_2(t) = \int -e^{2t} f(t)dt$.

This gives the y that we're looking for:

$$y(t) = v_1(t)e^{-t} + v_2(t)e^{-2t} = \left(\int e^t f(t)dt\right)e^{-t} + \left(\int -e^{2t} f(t)dt\right)e^{-2t}$$

(e) Why does this not contradict the theorem in linear algebra that says "If T is an onto linear transformation, then T is also one-to-one"?

The theorem only holds for **finite-dimensional** vector spaces! In this example, V is infinite-dimensional, because it contains $\{1, x, x^2, \cdots\}$, which is an infinite linearly independent subset of V.